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Second derivative in the model of classical binary system

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ABSTRACT

We have obtained an analytical expression for the second derivatives of the light curve with respect to geometric parameters in the model of eclipsing classical binary systems. These expressions are essentially efficient algorithm to calculate the numerical values of these second derivatives for all physical values of geometric parameters. Knowledge of the values of second derivatives of the light curve at some point provides additional information about asymptotical behaviour of the function near this point and can significantly improve the search for the best-fitting light curve through the use of second-order optimization method. We write the expression for the second derivatives in a form which is most compact and uniform for all values of the geometric parameters and so make it easy to write a computer program to calculate the values of these derivatives.

Key words: methods: analytical – methods: data analysis – methods: numerical – binaries: eclipsing.

1 INTRODUCTION

The task of finding light curve which is the best-fitting light to the observed values of the stellar flux is often performed using the model light curve with its first derivative (first-order methods). The best known of such methods is Gauss–Newton algorithm (Bates & Watt 1988) that is used to solve non-linear least-squares problem. This method is derived from application of the general Newton's method to the problem of finding the minimum residual if we disregard the term containing second derivatives of the light curve. Such disregard is justifiable if the residual value is small enough. However, this condition is not always satisfied, and if not, iterative sequence for Newton–Gauss algorithm converges slowly or does not converge at all. In particular, this trouble can arise if the initial values of the geometrical parameters are far from the minimum point or observational data contain significant errors. So, when the second derivatives (a second-order method) are used we can, in some cases, achieve or enhance the convergence. In spite the problem of calculation of the light curve and its first derivatives has been solved in different ways (Mandel & Agol 2002; Pal 2008; Pal 2012; Abubekerov & Gostev 2013), calculation of second derivatives has not yet been touched on.

In this work, we give analytical expressions for the second derivatives in the model of eclipsing classical binary systems regarding linear limb-darkening law and quadratic limb-darkening law. These expressions are written in terms of elementary functions, elliptical integrals and easily computable piecewise analytical functions of single variable, in a manner similar to that we used in our previous work (Abubekerov & Gostev 2013) for the light curve itself and its first derivatives. Using of these piecewise analytical functions, we can write the final expression uniformly for all values of geometrical parameters. To compute the elliptical integrals the efficient algorithms exist (Carlson 1995). We also give a general integral form of the second derivatives for limb-darkening law specified by arbitrary function. Therefore, this paper is a further development of the approach to the calculation of transit light curves represented by Abubekerov & Gostev (2013). We note that the usability of this method has since been confirmed using the results of that work for developing a software package PLANETPACK2 (Baluev 2014, the source code can be downloaded from http://sourceforge.net/projects/planetpack).

2 MODEL DESCRIPTION

We considered the model of the eclipse of a spherically symmetric star with thin atmosphere by another spherical opaque component (another spherical star or a spherical planet). We denote the radius of the eclipsed star as R_*, the radius of the eclipsing component as R_o, the distance between the centres of the discs of the components as D and the polar radius from the centre of the eclipsed star as ρ.

The brightness at the point of the disc of the eclipsed star with polar coordinate ρ is given by

$$J(\rho) = J(0) I\left(\frac{\rho}{R_*}\right).$$

Here $J(0)$ is the brightness at the centre of this stellar disc,

$$I(r) = (1 - f(\mu(r))),$$

$$\mu(r) = \sqrt{1 - r^2}$$

$$f(\mu) = \sum_k \Lambda_k f_k(\mu), \quad (1)$$

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where functions f_k are such that $f_k(1) = 0$, are defined by the law of limb-darkening in question, and Λ_k are the coefficients of limb-darkening.

In this paper, we consider the linear limb-darkening law, for which $f(\mu) = \Lambda_l f_l(\mu) = \Lambda_l(1 - \mu)$ and the quadratic law of limb-darkening, which is characterized by the presence of the term $\Lambda_q f_q(\mu) = \Lambda_q(1 - \mu)^2$ in the expression for f;

3 FORMULA FOR THE FLUX

The decrease of the flux of the binary system due to eclipse is (Abubekerov & Gostev 2013)

$$L^F - L(D, R_*, R_o) = \Delta L(D, R_*, R_o) = \iint\limits_{S(D)} J(|\boldsymbol{R}|)\,\mathrm{d}\boldsymbol{R} = J(0)R_*^2\Delta L(\delta, r),$$

where $r = \frac{R_o}{R_*}$, $\delta = \frac{D}{R_*}$, $S(D)$ is the area of overlapping discs. $\Delta L(\delta, r)$ can be written as

$$\Delta L(\delta, r) = \Delta L_0(\delta, r) + \Lambda_l \Delta L_l(\delta, r) + \Lambda_q \Delta L_q(\delta, r). \quad (2)$$

Thus, the current task is to find the second derivatives of $\Delta L(\delta, r)$ with respect to δ and r.

Let g be a function such that $g(\rho)$ is one of the linear terms in the expression for $I(\sqrt{\rho})$ [given by (1)], $g^{(-1)}$ is one of its primitives: $g(\rho) = \frac{\mathrm{d}g^{(-1)}(\rho)}{\mathrm{d}\rho}$. As it has been shown in Abubekerov & Gostev (2013), the contribution to $\Delta L(\delta, r)$ caused by the term $g(\rho)$ in the expression for $I(\sqrt{\rho})$ is

$$\iint\limits_{S(D)} g\left(\left|\frac{\boldsymbol{R}}{R_*}\right|\right)\mathrm{d}\boldsymbol{R} = \Delta L_g(\delta, r) = \Psi(\delta, 1, r)g^{(-1)}(1) - \pi\Theta(r - \delta)g^{(-1)}(0) + \int_0^{\Psi(\delta,r,1)} \frac{(r^2 - r\delta\cos(x))g^{(-1)}(\delta^2 + r^2 - 2r\delta\cos x)}{\delta^2 + r^2 - 2r\delta\cos x}\mathrm{d}x. \quad (3)$$

Its first derivatives is

$$\frac{\partial \Delta L_g(\delta, r)}{\partial \delta} = -2r \int_0^{\Psi(\delta,r,1)} \cos x\, g(\delta^2 + r^2 - 2r\delta\cos x)\mathrm{d}x \quad (4)$$

$$\frac{\partial \Delta L_g(\delta, r)}{\partial r} = 2r \int_0^{\Psi(\delta,r,1)} g(\delta^2 + r^2 - 2r\delta\cos x)\mathrm{d}x, \quad (5)$$

where

$$\Theta(t) \equiv \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0, \end{cases}$$

$$\mathcal{A}x \equiv \begin{cases} \pi, & x < -1 \\ \arccos x, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

and

$$\Psi(D, x, y) \equiv \mathcal{A}\left(\frac{x^2 + D^2 - y^2}{2\,x\,D}\right).$$

The expression for the decrease of the flux due to eclipse of the stellar disc with uniform brightness [for $\Delta L_0(\delta, r)$] can be obtained if we put $g(x) = 1$, $g^{-1}(x) = x$ in (3).

For the term with linear limb-darkening coefficients in (2)

$$\Delta L_l(\delta, r) = \Delta L_1(\delta, r) - \Delta L_0(\delta, r), \quad (6)$$

and for the term with quadratic limb-darkening coefficients in (2)[1]

$$\Delta L_q(\delta, r) = 2\Delta L_1(\delta, r) - 2\Delta L_0(\delta, r) + \Delta L_2(\delta, r). \quad (7)$$

Here, $\Delta L_1(\delta, r)$ is obtained, if we put $g(x) = \sqrt{1 - x}$, $g^{(-1)}(x) = -\frac{2}{3}(1 - x)^{3/2}$. Assuming $g(x) = x, g^{(-1)}(x) = x^2/2$ we get $\Delta L_2(\delta, r)$.

Therefore, now the main task is to find the second derivatives of $\Delta L_0(\delta, r)$, $\Delta L_1(\delta, r)$ and $\Delta L_2(\delta, r)$ with respect to r and δ. First, we find these derivatives in a general integral form [for $\Delta L_g(\delta, r)$].

4 GENERAL INTEGRAL FORMULAS FOR THE SECOND DERIVATIVES

Differentiating (4) and (5), we find

$$\frac{\partial^2 \Delta L_g(\delta, r)}{\partial \delta^2} = -4r \int_0^{\Psi(\delta,r,1)} \cos(x)(\delta - r\cos(x))\, g'(\delta^2 + r^2 - 2r\delta\cos x)\mathrm{d}x - \frac{2r}{\delta}\Phi(\delta, r)g(\delta^2 + r^2 - 2r\delta\Phi(\delta, r))(r^2 - \delta^2 - 1)\,\mathcal{Q}_1(\delta, r), \quad (8)$$

$$\frac{\partial^2 \Delta L_g(\delta, r)}{\partial \delta \partial r} = 4r \int_0^{\Psi(\delta,r,1)} (\delta - r\cos(x))\, g'(\delta^2 + r^2 - 2r\delta\cos x)\mathrm{d}x + \frac{2r}{\delta} g(\delta^2 + r^2 - 2r\delta\Phi(\delta, r))(r^2 - \delta^2 - 1)\,\mathcal{Q}_1(\delta, r), \quad (9)$$

$$\frac{\partial^2 \Delta L_g(\delta, r)}{\partial r^2} = \frac{1}{r}\frac{\partial \Delta L_g(\delta, r)}{\partial r} + 4r \int_0^{\Psi(\delta,r,1)} (r - \delta\cos(x))\, g'(\delta^2 + r^2 - 2r\delta\cos x)\mathrm{d}x + 2g(\delta^2 + r^2 - 2r\delta\Phi(\delta, r))(\delta^2 - r^2 - 1)\,\mathcal{Q}_1(\delta, r). \quad (10)$$

Here,

$$\mathcal{Q}x \equiv \begin{cases} \sqrt{x}, & x \ge 0 \\ 0, & x < 0, \end{cases}$$

[1] See Appendix B.

$$Q_1(\delta, r) \equiv \mathcal{Q}\left(\frac{1}{\left(1-(\delta-r)^2\right)\left((\delta+r)^2-1\right)}\right).$$

$$\Phi\,(\delta, r) \equiv \cos\Psi(\delta, r, 1) \equiv \vartheta\left(\frac{\delta^2+r^2-1}{2\,\delta\,r}\right),$$

$$\vartheta(x) \equiv \cos\mathcal{A}\,x \equiv \begin{cases} -1, & x<-1 \\ x, & -1\le x\le 1 \\ 1, & x>1. \end{cases}$$

5 THE SECOND DERIVATIVES FOR INDIVIDUAL LAWS OF LIMB-DARKENING

The expression for $\frac{\partial\Delta L_g(\delta,r)}{\partial r}$, which occurs in (10), was calculated on the assumption of individual laws of limb-darkening in Abubekerov & Gostev (2013).

Putting $g(x) = 1$ in (8)–(10) for zero case (the decrease of the flux due to the eclipse by the uniform brightness disc), we get

$$\frac{\partial^2\Delta L_0(\delta, r)}{\partial\delta^2} = -\frac{2r}{\delta}\Phi(\delta, r)\left(r^2-\delta^2-1\right)Q_1(\delta, r)$$
$$= \frac{Q(\delta, r)}{\delta^2} - 2\left(1+r^2-\delta^2\right)Q_1(\delta, r), \qquad (11)$$

$$\frac{\partial^2\Delta L_0(\delta, r)}{\partial\delta\partial r} = -\frac{2r}{\delta}\left(1-r^2+\delta^2\right)Q_1(\delta, r), \qquad (12)$$

and

$$\frac{\partial^2\Delta L_0(\delta, r)}{\partial r^2} = 2\Psi(\delta, r, 1) - 2\left(1+r^2-\delta^2\right)Q_1(\delta, r). \qquad (13)$$

Here,

$$Q\,(\delta, r) \equiv \mathcal{Q}\left(\left(1-(\delta-r)^2\right)\left((\delta+r)^2-1\right)\right).$$

Putting $g(x) = \sqrt{1-x}$ in (8)–(10), we find that the second derivatives of ΔL_1 are

$$\frac{\partial^2\Delta L_1(\delta, r)}{\partial\delta^2}$$
$$= \frac{\mathcal{Q}\left(1-(r-\delta)^2\right)(2r^2-4\delta^2-2)\hat{E}(\delta, r)}{3\delta^2}$$
$$+\ \mathcal{Q}\left(\frac{1}{1-(r-\delta)^2}\right)\frac{1}{3\delta^2}$$
$$\times\left[\left(1-(r-\delta)^2\right)\left((r+\delta)^2-1\right)\right.$$
$$\left.+\ 3\left(r^2-\delta^2-1\right)\left(r^2+\delta^2-1\right)\right]\hat{F}(\delta, r)$$
$$-\frac{2r}{\delta}\Phi(\delta, r)\sqrt{1-\delta^2-r^2+2r\delta\Phi(\delta, r)}$$
$$\times(r^2-\delta^2-1)\,Q_1(\delta, r), \qquad (14)$$

$$\frac{\partial^2\Delta L_1(\delta, r)}{\partial\delta\partial r}$$
$$= \frac{2r}{\delta}\,\mathcal{Q}\left(1-(r-\delta)^2\right)\hat{E}(\delta, r)$$
$$+\ \frac{2r(r^2-\delta^2-1)}{\delta}\,\mathcal{Q}\left(\frac{1}{1-(r-\delta)^2}\right)\hat{F}(\delta, r)$$
$$+\ \frac{2r}{\delta}\sqrt{1-\delta^2-r^2+2r\delta\Phi(\delta, r)}\left(r^2-\delta^2-1\right)Q_1(\delta, r), \qquad (15)$$

$$\frac{\partial^2\Delta L_1(\delta, r)}{\partial r^2} = 6\,\mathcal{Q}\left(1-(r-\delta)^2\right)\hat{E}(\delta, r)$$
$$+2\left(\delta^2-r^2-1\right)\mathcal{Q}\left(\frac{1}{1-(r-\delta)^2}\right)\hat{F}(\delta, r)$$
$$+2\sqrt{1-\delta^2-r^2+2r\delta\Phi(\delta, r)}\left(\delta^2-r^2-1\right)Q_1(\delta, r). \qquad (16)$$

Here,[2]

$$\hat{F}(\delta, r) \equiv F\left(\frac{\Psi(\delta, r, 1)}{2}\,\middle|\,\frac{4\delta r}{1-(r-\delta)^2}\right) \qquad (17)$$

$$\hat{E}(\delta, r) \equiv E\left(\frac{\Psi(\delta, r, 1)}{2}\,\middle|\,\frac{4\delta r}{1-(r-\delta)^2}\right), \qquad (18)$$

where F and E are incomplete elliptic integrals of the first and second kind:

$$F(\phi\,|m) \equiv \int_0^{\phi}\frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2(\theta)}},$$

$$E(\phi\,|m) \equiv \int_0^{\phi}\sqrt{1-m\sin^2(\theta)}.$$

Next, we write the second derivatives of ΔL_2. Here, it is useful to note that

$$\sin\Psi(\delta, r, 1) \equiv \sqrt{1-\Phi(\delta, r)^2} \equiv \frac{Q(\delta, r)}{2\delta r}.$$

Putting $g(x) = x$ in (8)–(10), we get

$$\frac{\partial^2\Delta L_2(\delta, r)}{\partial\delta^2}$$
$$= \frac{(r\Phi(\delta, r)-2\delta)Q(\delta, r)}{\delta} + 2r^2\Psi(\delta, r, 1)$$
$$-\frac{2r}{\delta}\Phi(\delta, r)(\delta^2+r^2-2r\delta\Phi(\delta, r))(r^2-\delta^2-1)\,Q_1(\delta, r), \qquad (19)$$

$$\frac{\partial^2\Delta L_2(\delta, r)}{\partial\delta\partial r} = 4\delta r\Psi(\delta, r, 1) - \frac{2r\,Q(\delta, r)}{\delta}$$
$$+\ \frac{2r}{\delta}(\delta^2+r^2-2r\delta\Phi(\delta, r))(r^2-\delta^2-1)\,Q_1(\delta, r), \qquad (20)$$

$$\frac{\partial^2\Delta L_1(\delta, r)}{\partial r^2} = 2(\delta^2+3r^2)\Psi(\delta, r, 1) - 4Q(\delta, r)$$
$$+2(\delta^2+r^2-2r\delta\Phi(\delta, r))(\delta^2-r^2-1)\,Q_1(\delta, r). \qquad (21)$$

[2] See Appendix A.

6 CONCLUSION

Thus, we have obtained analytical expressions for the second derivatives of the flux with respect to geometric parameters, uniform for all values of parameters. We considered the linear and quadratic limb-darkening laws. Directly, second derivatives have been calculated of the flux decrease due to eclipse that is given by (2), (6) [for the component caused by linear law of limb-darkening] and (7) [for the component caused by quadratic law of limb-darkening]. The second derivatives of ΔL_0 (the component caused by uniform brightness disc) are given in (11)–(13). The second derivatives of ΔL_1 are given in (14)–(16). The second derivatives of ΔL_2 are given in (19)–(21).

The current expressions seem to be longer and more complicated than the ones for the first derivatives in (Abubekerov & Gostev (2013)), but parts of the expressions, the calculation of which takes most of the time are the same. Each part can be computed once at the same values of the parameters, so we don't get any significant increase in computation time if compared to using only first derivatives. Furthermore, as noted above, the employed approach to calculation of second derivatives of the light curve is a direct continuation of the approach developed in Abubekerov & Gostev (2013). For this reason, someone who has already worked with the software implementation of the algorithm from Abubekerov & Gostev (2013) can effectively use the previous results for easily implement computation of the second derivatives in line with this paper. As mentioned above, the results of Abubekerov & Gostev (2013) have been used in Baluev (2014).

The algorithm is implemented in ANSI C in the form of functions for computation of the individual component $\Delta L(\delta, r)$ and its derivatives. This implementation is available at http://lnfm1.sai.msu.su/~ngostev/algorithm.html.

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APPENDIX A

If $\Psi(\delta, r, 1) < \pi$, incomplete elliptic integrals in (17) and (18) match its special cases:

$$F\left(\arcsin\left(\frac{1}{\sqrt{x}}\right)\middle|\, x\right) = \frac{1}{\sqrt{x}} K\left(\frac{1}{x}\right),$$

$$E\left(\arcsin\left(\frac{1}{\sqrt{x}}\right)\middle|\, x\right) = \sqrt{x}\left[E\left(\frac{1}{x}\right) + \left(\frac{1}{x} - 1\right) K\left(\frac{1}{x}\right)\right],$$

where $K(x) = F\left(\frac{\pi}{2}\middle|\, x\right) x$, $E(x) = E\left(\frac{\pi}{2}\middle|\, x\right)$ are *complete* elliptic integrals of the first and second kind, respectively. Thus, in the mentioned case

$$\hat{F}(\delta, r) = \sqrt{\frac{1-(r-\delta)^2}{4\delta r}}\, K\left(\frac{1-(r-\delta)^2}{4\delta r}\right), \quad \text{(A1)}$$

$$\hat{E}(\delta, r) = \sqrt{\frac{4\delta r}{1-(r-\delta)^2}}\left[E\left(\frac{1-(r-\delta)^2}{4\delta r}\right) + \frac{1-(r+\delta)^2}{4\delta r} K\left(\frac{1-(r-\delta)^2}{4\delta r}\right)\right]. \quad \text{(A2)}$$

This remark may be useful to optimize the computing. Moreover, if we replace the right parts of (A1) and (A2) by its real parts, the resulting expressions are valid for all values of δ and r. It should be noted that the algorithms which are described in the Carlson (1995) are applicable for computing the complex values of the complete elliptic integrals.

APPENDIX B

It is noticed the erratum in our previous work (Abubekerov & Gostev 2013). Wrong version of formula (29):

$$\Delta L_q(\delta, r) = 2\Delta L_1(\delta, r) - \Delta L_0(\delta, r) - \Delta L_2(\delta, r),$$

$$L_q^f = 2L_1^f - L_0^f - L_2^f = \frac{\pi}{6}.$$

Correct version:

$$\Delta L_q(\delta, r) = 2\Delta L_1(\delta, r) - 2\Delta L_0(\delta, r) + \Delta L_2(\delta, r),$$

$$L_q^f = 2L_1^f - 2L_0^f + L_2^f = -\frac{\pi}{6}.$$

It does not affect the rest.

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